Short communication regarding the paper

Random effects Weibull regression model for occupational lifetime

European Journal of Operational Research 179, 124 - 131

Stochastics and Statistics

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With great interest we read the recent publication of Sohn et al. (2007) on a Weibull regression model with random effects for modelling occupational lifetime. We congratulate Sohn and colleagues on their comprehensive and clearly written paper. Nevertheless, we would like to comment on two points, the first regarding the moments of the underlying Weibull distribution, the second regarding the relations of Sohn’s model to the class of frailty models.

For the Weibull distribution the following parametrization is used by Sohn et al. in their formula (2) (where we drop the index \( i \) for ease of presentation):

\[
f(t|\theta, \beta) = \frac{\beta}{\theta} t^{\beta-1} \exp\left(-\frac{t^\beta}{\theta}\right) \quad (1)
\]

Unfortunately, the conditional mean and variance given in formula (3) of Sohn et al. are incorrect. The correct expressions are

\[
E(t|\theta, \beta) = \Gamma(1 + \frac{1}{\beta}) \theta^{\frac{1}{\beta}} \quad \text{and}
\]

\[
V(t|\theta, \beta) = \left(\Gamma(1 + \frac{2}{\beta}) - \Gamma(1 + \frac{1}{\beta})^2\right) \theta^2.
\]

In the course of the paper, these expressions are used in formula (10), rendering expectation and variance of the (unconditional) occupational lifetime also incorrect. The correct expressions are derived in the following, also based on the assumption made by Sohn et al. that the random effect \( \theta \) is inverse Gamma distributed with \( \theta \sim \Gamma\left(\alpha + 1, \alpha \exp(XB)\right) \).

First, straightforward calculations yield the general moments for the inverse Gamma distribution \( Y \sim \Gamma(a, b) \) with density

\[
f(y) = \frac{b^a}{\Gamma(a)} y^{-(a+1)} e^{-\frac{b}{y}}
\]

by

\[
EY^\gamma = \frac{b^{\gamma} \Gamma(a - \gamma)}{\Gamma(a)}.
\]

This allows us to calculate the conditional expectation of time as:

\[
E_t = E(E(t|\theta, \beta)) = \Gamma(1 + \frac{1}{\beta}) \mathbb{E}\theta^{\frac{1}{\beta}} = \Gamma(1 + \frac{1}{\beta}) \left(\alpha \exp(XB)\right)^{\frac{1}{\beta}} \frac{\Gamma(\alpha + 1 - \frac{1}{\beta})}{\Gamma(\alpha + 1)}.
\]
For the unconditional variance this becomes:

\[ \mathbf{V}(t) = \mathbf{E}(\mathbf{V}(t|\theta, \beta)) + \mathbf{V}(\mathbf{E}(t|\theta, \beta)) \]

\[ = \left( \Gamma(1 + \frac{1}{2\beta}) - \Gamma(1 + \frac{1}{3\beta})^2 \right) \mathbf{E}(\theta^{\frac{3}{2}}) \]
\[ + \Gamma(1 + \frac{1}{3\beta})^2 (\mathbf{E}(\theta^{\frac{3}{2}})) - (\mathbf{E}(\theta^{\frac{3}{2}}))^2 \]
\[ = \left( \Gamma(1 + \frac{1}{2\beta}) - \Gamma(1 + \frac{1}{3\beta})^2 \right) \frac{(\alpha \exp(XB))^\frac{3}{2} \Gamma(\alpha + 1 - \frac{2}{3\beta})}{\Gamma(\alpha + 1)} \]
\[ + \Gamma(1 + \frac{1}{3\beta})^2 \left[ \frac{(\alpha \exp(XB))^\frac{3}{2} \Gamma(\alpha + 1 - \frac{2}{3\beta})}{\Gamma(\alpha + 1)} - \left( \frac{(\alpha \exp(XB))^\frac{3}{2} \Gamma(\alpha + 1 - \frac{1}{3\beta})}{\Gamma(\alpha + 1)} \right)^2 \right] \]
\[ = \Gamma(1 + \frac{1}{2\beta}) \frac{(\alpha \exp(XB))^\frac{3}{2} \Gamma(\alpha + 1 - \frac{2}{3\beta})}{\Gamma(\alpha + 1)} \]
\[ - \Gamma(1 + \frac{1}{3\beta})^2 \left( \frac{(\alpha \exp(XB))^\frac{3}{2} \Gamma(\alpha + 1 - \frac{1}{3\beta})}{\Gamma(\alpha + 1)} \right)^2 . \]

Luckily, these formulas for the conditional expectation and variance are the only ones in the Sohn et al. paper that are influenced by the computational errors in the moments. All other computations in the paper and also those for the application data remain valid.

Let us consider next the link between the suggested model by Sohn et al. (2007) and frailty models. We consider the following re-parametrization of the Weibull model (1):

\[ f(t|\theta^*, \beta) = \beta \theta^* t^{\beta-1} e^{-\theta^* t^\beta}, \quad (2) \]

where we set \( \theta^* = 1/\theta \), as compared to the original form of Song et al. The parameter \( \theta^* \) now has a Gamma (instead of an inverse Gamma) distribution, and the model can be written in a similar fashion as \( \theta^* = Z \exp(XB) \) with \( Z \sim \Gamma(\frac{1}{\sigma^2}, \frac{1}{\sigma^2}) \). The parameter \( \sigma^2 \) denotes the variance of the random effect (frailty) \( Z \), the expectation of \( Z \) being 1. Please note the subtle differences in the way the covariates (\( \exp(XB) \)) are included into the model equations.

Gamma frailty models are a common tool in survival analysis with random effects, dating back at least to Beard (1959), Vaupel et al. (1979) and Lancaster (1979) and are considered in more detail in the biostatistical literature.
In general, the unconditional survival function can be derived by using the Laplace transform of the Gamma distribution and the baseline hazard function in a very elegant way:

\[ S(t) = \mathbb{E}S(t|\theta^*, \beta) = \mathbb{E}e^{-\Lambda(t|\theta^*, \beta)}, \]

where \( \Lambda(t|\theta^*, \beta) = \int_0^t \lambda_0(s|\theta^*, \beta) \, ds \) describes the cumulative baseline hazard function. Under the common assumption of a proportional hazards model with random effects \( \lambda(t|\theta^*, \beta) = Z\lambda_0(t) \exp(XB) \) with \( Z \sim \Gamma(\frac{1}{\sigma^2}, \frac{1}{\sigma^2}) \) it holds

\[
S(t) = \mathbb{E}e^{-\Lambda(t|\theta^*, \beta)} \\
= \mathbb{E}e^{-Z\Lambda_0(t) \exp(XB)} \\
= L_Z(\Lambda_0(t) \exp(XB)) \\
= (1 + \sigma^2 \Lambda_0(t) \exp(BX))^{-\frac{1}{\sigma^2}}.
\]

Note that this derivation only uses properties of the Gamma distribution and elementary relations in survival models. Especially, \( \Lambda_0(t) \) can be any cumulative baseline hazard function.

That is, the Weibull model of Sohn et al. is a special case of frailty models with \( \Lambda_0(t) = \theta^* t^\beta \) and closed form likelihoods can be given for various other survival distributions. It is not even necessary to parameterize the baseline hazard function by Weibull, even with non-parametric baseline hazards the model parameters are still identifiable if covariates are included the model (Elbers and Ridder 1982). Several other generalizations depending on the given problem at hand are possible, but are beyond the scope of this short communication. We refer the interested reader to the books by Hougaard (2000) and Duchateau and Janssen (2008).

Admittedly, it is not necessary to formulate the model of Sohn et al. in the context of frailty models, the model can stand for its own as described by Sohn et al. However, understanding Sohn’s model as a special case of frailty models offers a bundle of new options for model generalization, the most important point being a wealth of additional survival distributions. And as an additional
benefit, this advantage comes at virtually no cost, as closed form likelihoods and moments (as they are given in Sohn’s model) are also available for other gamma frailty models.

References


